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Mh4718 Week 12

Week 12

0.1 Fixed Point Iteration (contd.)

0.1.0.1 Solving Equations.
We can write any equation to be solved in the form $f(x) = x$ and to solve we check the criteria for convergence e.g. quadratics.

Example 0.1

Solve $x^2 - 5x + 6 = 0$ by fixed-point iteration.
First we write this equation as a fixed point equation. There are many ways that this can be done. The most obvious thing to do is simpy add x to both sides of the equation giving the equation:

$$x^2 - 4x + 6 = x.$$

We can attempt to solve the equation $x^2 - 5x + 6 = 0$ by finding a fixed point of $F(x) = x^2 - 4x + 6$ by iteration.
We can check whether we have guaranteed convergence or not for the iteration since we know that the solutions to $x^2 - 5x + 6 = 0$ are $x = 2$ and $x = 3$.

Note that $F'(x) = 2x - 4$ and $|F'(2)| = 0, |F'(3)| = 2$. According the above theory then we are guaranteed convergence to the fixed point 2 if we pick x_0 "close enough" to 2.

We can also write the equation $x^2 - 5x + 6 = 0$ as

- $x = \dfrac{x^2+6}{5}$
- $x = 5 - \dfrac{6}{x}$
- $x = \sqrt{5x-6}$.

That is we can have

- $F(x) = \dfrac{x^2+6}{5}$
- $F(x) = 5 - \dfrac{6}{x}$
- $F(x) = \sqrt{5x-6}$

Let us analyse the convergence behaviour when we iterate each of these respecively:

- $$F(x) = \frac{x^2+6}{5} \Rightarrow F'(x) = \frac{2x}{5} \Rightarrow |F'(2)| = \frac{4}{5}(< 1) \text{ and } |F'(3)| = \frac{6}{5}(> 1).$$

  Therefore we are guaranteed that iterating $\dfrac{x^2+6}{5}$ will converge to 2 if we pick x_0 "close enough" to 2.

- $$F(x) = 5 - \frac{6}{x} \Rightarrow F'(x) = \frac{6}{x^2} \Rightarrow |F'(2)| = \frac{6}{4}(> 1) \text{ and } |F'(3)| = \frac{6}{9}(< 1).$$

  Therefore we are guaranteed that iterating $5 - \dfrac{6}{x}$ will converge to 3 if we pick x_0 "close enough" to 3.

- $$F(x) = \sqrt{5x-6} \Rightarrow F'(x) = \frac{\frac{5}{2}}{\sqrt{5x-6}} \Rightarrow |F'(2)| = \frac{\frac{5}{2}}{\sqrt{4}} = \frac{5}{4}(> 1) \text{ and } |F'(3)| = \frac{\frac{5}{2}}{\sqrt{9}} = \frac{5}{6}(< 1).$$

  Therefore we are guaranteed that iterating $\sqrt{5x-6}$ will converge to 3 if we pick x_0 "close enough" to 3.

Recall that the *Newton Raphson* method iterates the function $F(x) = x - \dfrac{f(x)}{f'(x)}$ in order to solve the equation $f(x) = 0$ because

$$F(p) = p \Rightarrow f(p) = 0.$$

In order for this iteration to work we must have $f'(x) \neq 0$ around p.

What about local convergence for the Newton Raphson method? To answer this question note that

$$F'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

therefore $F'(p) = 0$ (since $f(p) = 0$) and so F is locally convergent to each of its fixed points p.

0.1.1 Solving Linear equations.

There are iterative schemes for solving systems of linear equations.

Example 0.2

Solve the equations

$$3x + y - z = 3$$
$$x - 4y + 2z = -1$$
$$-2x - y + 5z = 2$$

As we know such a set of equations can be written using matrix and vector notation.

Example 0.3

The equations

$$3x + y - z = 3$$
$$x - 4y + 2z = -1$$
$$-2x - y + 5z = 2$$

can be written as:

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & -4 & 2 \\ -2 & -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

In general a system of $n$linear equations in n unknowns can thus be written as

$$\mathbf{A}\mathbf{v} = \mathbf{b}$$

where $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{v}$ is an n-dimensional vector and $\mathbf{b}$ is an n-dimentional vector.
Such a set of equations can also be written as a fixed point equation.

Example 0.4

The equations

$$\begin{aligned} 3x + y - z &= 3 \\ x - 4y + 2z &= -1 \\ -2x - y + 5z &= 2 \end{aligned}$$

can be written as:

$$\begin{aligned} 3x &= -y + z + 3 \\ -4y &= -x - 2z - 1 \\ 5z &= 2x + y + 2 \end{aligned}$$

That is

$$\begin{aligned} x &= -\frac{1}{3}y + \frac{1}{3}z + 1 \\ y &= \frac{1}{4}x + \frac{1}{2}z + \frac{1}{4} \\ z &= \frac{2}{5}x + \frac{1}{5}y + \frac{2}{5} \end{aligned}$$

or in matrix notation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{2}{5} & \frac{1}{5} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{1}{4} \\ \frac{2}{5} \end{pmatrix}$$

In general a system of n equations in n unknowns can be written as a fixed point equation of the form

$$\mathbf{v} = \mathbf{T}\mathbf{v} + \mathbf{c}$$

In many cases solutions to the fixed point version of the linear system can be estimated at by an iteration process.
One such process is known as **Jacobi** iteration.
If we wish to solve the system

$$\mathbf{Av} = \mathbf{b}$$

where

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

We solve the first line of the equations for x_1 in terms of all the other variables, the second line for x_2 in terms of all the other variables and so on and transform the system into a fixed point equation

$$\mathbf{v} = \mathbf{Tv} + \mathbf{c}$$

We then pick an initial guess

$$\mathbf{v^{(0)}} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ . \\ . \\ . \\ x_n^{(0)} \end{pmatrix}$$

and proceed to compute a sequence of vectors by iteration

$$\mathbf{v^{(n+1)}} = \mathbf{Tv^{(n)}} + \mathbf{c}, n = 0, 1, 2 \ldots$$

If the original matrix **A** is diagonally dominant, then this iteration will converge to a fixed point which supplies solutions to the system.

A matrix is diagonally dominant if each diagonal element has an absolute value which is greater than the sum of the absolute values of all the off-diagonal elements in the same row

It is easy to check that the system

$$3x + y - z = 3$$

$$x - 4y + 2z = -1$$

$$-2x - y + 5z = 2$$

has solution $x = 1, y = 1, z = 1$ and the matrix

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & -4 & 2 \\ -2 & -1 & 5 \end{pmatrix}$$

is diagonally dominant.